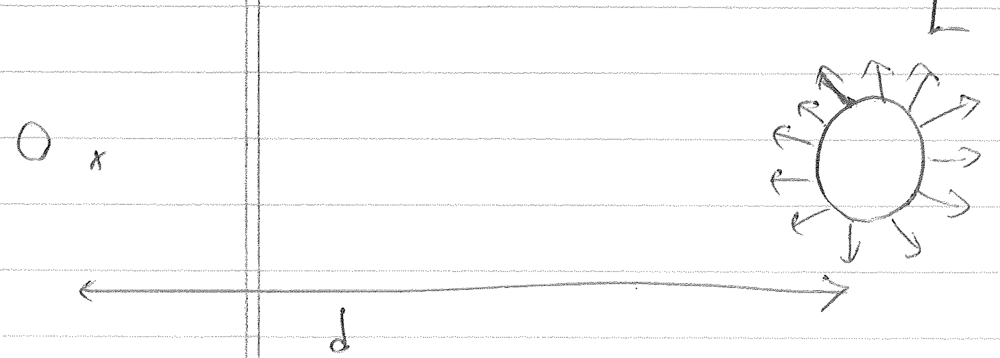
Lec 5: 09/11/2013

Luminosity Distances and Deceleration Parameter:

Consider an object with absolute luminosity L (defined as energy radiated per second), which is at a distance d from the observer.



The apparent luminosity ℓ according to the observer (defined as energy radiated per second per unit area) is given by:

$$\ell = \frac{L}{4\pi d^2}$$

If L is known for an object (the case for "standard candles") and ℓ is measured, then d can be found from the above relation.

The physical distance is related to the absolute and apparent luminosities according to $d = \left(\frac{L}{4\pi \ell}\right)^{1/2}$.

In an expanding universe there are additional effects that must be taken into account. First, energy radiated per unit time will be smaller by a factor of $(1+z)$ for the observer because of the frequency redshift in an expanding universe. Second, energy of individual photons will also be redshifted by the same factor because of the momentum redshift in an expanding universe. Including both of the effects, we find:

$$\ell = \frac{L}{4\pi d^2 (1+z)^2}$$

One can define a luminosity distance d_L such that $\ell = \frac{L}{4\pi d_L^2}$

Luminosity distance is related to physical distance according to

$$d_L = d\,(1+z)$$

In an expanding universe, the physical distance between the observer and the source at the time of detection t_0 is $a_{(t_0)} r_1$, (e; hen

(comoving coordinate of the source)

$$d_L = a_{(t_0)}\, r_1\, (1+z)$$

So far, we have made an implicit assumption that the universe is geometrically flat ($k=0$), or we look at nearby objects for which the universe is locally flat regardless of k. The point is that to find l we have divided L by the area of a sphere of radius d. Depending on the geometry, the area has a differend dependence on d:

$$S = 4\pi \times \begin{cases} r^2 & k=0 \\ \sin^2 r & k=+1 \\ \sinh^2 r & k=-1 \end{cases}$$

Therefore, the precise definition of luminosity distance is

$$d_L = R(r)\, a(t_0)\,(1+z) \qquad R(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$$

One can calculate r_1 from the following:

$$r_1 = \int_{t}^{t_0} \frac{dt'}{a(t')}$$

Knowing the energy content of the universe, one can solve the first Friedmann equation to find $a(t')$, and then perform the above integral to find r. By making the following change of variable, we can find a more conveniend expression for r_1 in a general case:

$$x \equiv \frac{a}{a_0} \Rightarrow dx = \frac{\dot{a}}{a_0}\, dt = \frac{\dot{a}}{a}\, \overbrace{\frac{a}{a_0}}^{x}\, dt \Rightarrow dt = \frac{dx}{x H} \quad (\mathrm{I})$$

From the first Friedmann equation:

$$H^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2} \Rightarrow H = \left(\frac{8\pi G \rho}{3} - \frac{k}{a^2}\right)^{\frac{1}{2}}$$

Let us include various contributions to the energy density ρ and define $\rho_k = -\frac{3k}{8\pi G a^2}$ to be the energy density associated with the curvature:

$$\rho_{tot} = \rho_r + \rho_m + \rho_v + \rho_k \quad , \quad H = \left(\frac{8\pi G}{3}\, \rho_{tot}\right)^{\frac{1}{2}} \quad (\mathrm{II})$$

Considering how different components of ρ depend on a, we have:

$$\rho_{tot} = \rho_{r_0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{m_0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{v_0} + \rho_{k_0}\left(\frac{a}{a_0}\right)^{-2} \Rightarrow \rho_{tot} = \rho_{r_0}\, x^{-4} +$$

$$\rho_{m_0}\, x^{-3} + \rho_{v_0} + \rho_{k_0}\, x^{-2} \quad (\mathrm{III})$$

From Eqs. (I, II, III) we find:

$$dt = \frac{da}{a\left(\frac{8\pi G}{3}\right)^{1/2}\left[\rho_{v_0} + \rho_{r_0} a^{-4} + \rho_{m_0} a^{-3} + \rho_{k_0} a^{-2}\right]^{1/2}}$$

Thus:

$$r_1 = \int_t^{t_0} \frac{dt'}{a(t')} = \frac{1}{a_0}\left(\frac{3}{8\pi G}\right)^{\frac{1}{2}} \int_{\frac{1}{1+z}}^{1} \frac{da}{a^2\left[\rho_{v_0} + \rho_{r_0} a^{-4} + \rho_{m_0} a^{-3} + \rho_{k_0} a^{-2}\right]^{1/2}}$$

One can normalize the energy density of different components with the current value of the Hubble rate:

$$H_0 = \left(\frac{8\pi G}{3}\right)^{1/2} \rho_0^{\frac{1}{2}}$$

$$\Omega_{\Lambda_0} \equiv \frac{\rho_{v_0}}{\frac{3}{8\pi G} H_0^2}, \quad \Omega_{r_0} \equiv \frac{\rho_{r_0}}{\frac{3}{8\pi G} H_0^2}, \quad \Omega_{m_0} \equiv \frac{\rho_{m_0}}{\frac{3}{8\pi G} H_0^2}, \quad \Omega_{k_0} \equiv \frac{\rho_{k_0}}{\frac{3}{8\pi G} H_0^2}$$

Note that by definition $\Omega_{\Lambda_0} + \Omega_{r_0} + \Omega_{m_0} + \Omega_{k_0} = 1$. The values of $\Omega_{\Lambda_0}, \Omega_{m_0}, \Omega_{k_0}$ within the standard model of cosmology are inferred from the CMB data:

$$\Omega_{\Lambda_0} \simeq 0.68, \quad \Omega_{m_0} \simeq 0.32, \quad \Omega_{k_0} \sim 1\%$$

From the temperature of CMB $T \simeq 2.72\,^\circ K$ we can find

$\Omega_{r_0} \sim 10^{-4}$, which is totally negligible.

Hubble Expansion and Deceleration Parameter:

The deceleration parameter q is defined as:

$$q \equiv -\frac{\ddot{a}}{a}\frac{1}{H^2}$$

It tells us how the expansion behaves: if $q>0$, then we have decelerating expansion, while for $q<0$ the expansion is accelerating.

The Hubble rate and the deceleration parameter can be used to obtain the redshift z as a function of the look-back time of an object:

$$z = \frac{a(t_0)}{a(t)} - 1 \qquad a(t) = a_0 + \dot{a}_0 (t-t_0) + \frac{1}{2}\ddot{a}_0 (t-t_0)^2 + \dots$$

$$a(t) = a(t_0)\left[1 + \left(\frac{\dot{a}}{a}\right)_0 (t-t_0) + \frac{1}{2}\left(\frac{\ddot{a}}{a}\right)_0 (t-t_0)^2 + \dots\right] = a_0\left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots\right]$$

Then:

$$\frac{1}{a(t)} = \frac{1}{a(t_0)}\Big\{\Big[1 - [H_0(t-t_0) - \frac{1}{2}q_0H_0^2(t-t_0)^2 + \ldots] + [H_0(t-t_0) + \frac{1}{2}q_0H_0^2(t-t_0)^2 + \ldots]^2$$

$$+ \ldots\Big]\Big\} \Rightarrow z = \frac{a_0}{a(t)} - 1 = H_0(t_0 - t) + \frac{1}{2}(q_0+2)H_0^2(t_0-t)^2 + \ldots$$

The Taylor's series expansion of $a(t)$ can also be used to find the comoving coordinate of the object as a function of its redshift:

$$r_1 = \int_t^{t_0} \frac{dt'}{a(t')} = \frac{1}{a(t_0)}\int_t^{t_0}[1 + H_0(t_0 - t') + \frac{1}{2}(q_0+2)H_0^2(t_0-t')^2 + \ldots]\,dt'$$

$$\Rightarrow r_1 a_{(t_0)} = (t_0 - t) + \frac{1}{2}H_0(t_0-t)^2 + \ldots$$

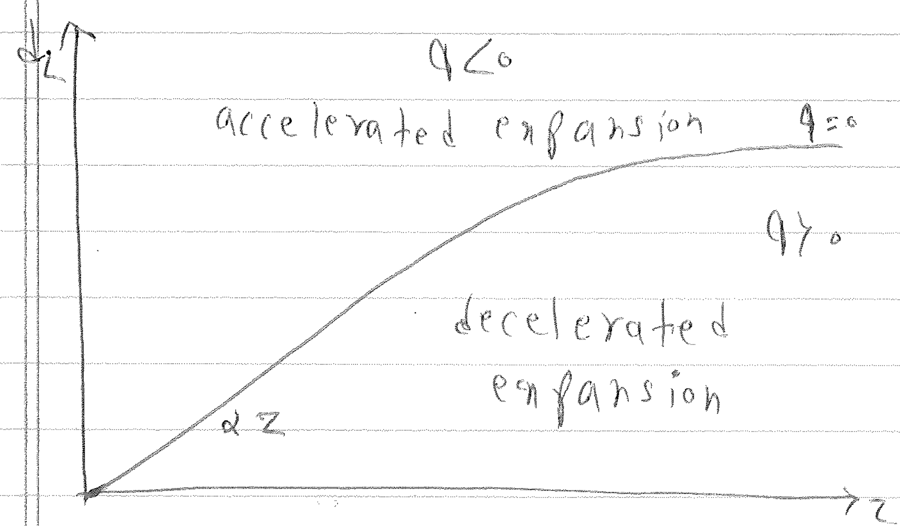
In terms of z, this can be rewritten to find:

$$r_1 a_{(t_0)} = \left[z - \frac{1}{2}(1+q_0)z^2\right]H_0^{-1}$$

Therefore, the luminosity distance as a function of is found to be:

$$d_L = r_1 a_{(t_0)}(1+z) = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2 + \ldots\right] \quad (\text{IV})$$

By measuring luminosity distance of objects that are standard candles as a function of their redshift, we can find the Hubble expansion rate and the deceleration parameter at the present time (H_0 and q_0 respectively):



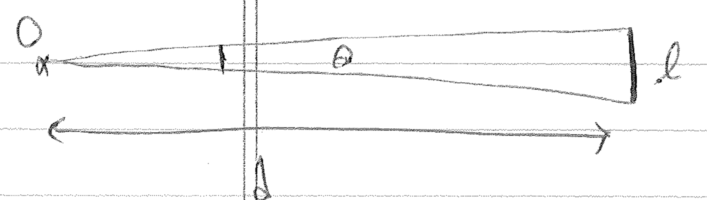
At very small redshifts only the $H_0^{-1} z$ term is important, through which we can find H_0. At higher redshifts, the quadratic term become significant, which can be used to find q_0. The data from type Ia supernova at high redshift indicated $q_0 < 0$, hence accelerated expansion at the present time.

For a universe that contains only vacuum energy, one can find

an exact expression for luminosity distance $d_L = H_0^{-1}(z + z^2)$.
Therefore the coefficient of the z^2 term should be $\simeq 1$ for those type Ia supernova at high z. At even higher redshift, however, matter was dominant over vacuum energy in our universe. This is because $\rho_v = $ const., while $\rho_m \propto a^{-3}_{(t)}$. Hence, since $\frac{\rho_{v_0}}{\rho_{m_0}} \sim \frac{68}{32}$, matter was dominant earlier on (roughly before 10^{10} yr). We there expect a switch from accelerated expansion ($q_0 < 0$) to decelerated expansion ($q_0 > 0$) at some high redshift. This has been observed in the data and confirms the so-called Λ-CDM model (Λ for cosmological constant, or vacuum energy, and CDM for cold dark matter).

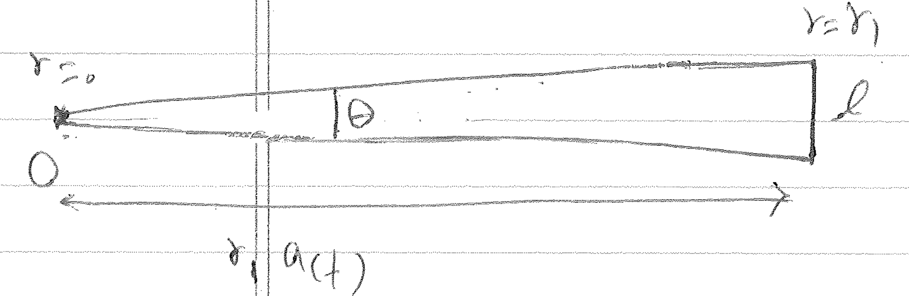
<u>Angular Diameter Distance:</u>

One may also use "standard rulers" in addition to "standard candles" to obtain parameters related to the expansion of the universe. Standard rulers are object whose size is known. Consider such an object at a distance from the observer as follows:



$$\theta \simeq \frac{l}{d}$$

For a nearby object in an expanding universe, the geometry is essentially flat. The angular diameter distance can then be defined to conform with the above expression:



$$d_A \equiv \frac{l}{\theta} = r_1\, a(t)$$

Here "t" denotes the time at which the photons left the object. We see from Eq. (IV) that $d_A = d_L\,(1+z)^{-2}$, which results in:

$$d_A = H_0^{-1}\left[z - \frac{1}{2}(3+q_0)\, z^2\right] \qquad (\mathrm{V})$$

Note that $d_A = d_L$ in a static universe.

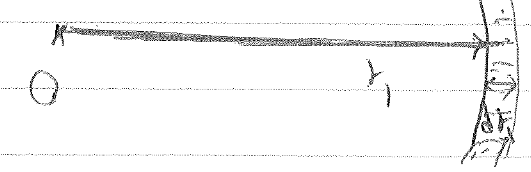
A potential problem with standard rulers is that we need systems with a well-defined and well-know size. Of course, objects like galaxies and supernova do not have well-defined edges. Therefore angular diameter distances are much less useful in studying the cosmological expansion than luminosity distances are.

However, they play an important role in theoretical analysis of CMB and large scale structure (LSS). As we will see later, a precise standard ruler exists in the early universe, which we can use to gain valuable information about our universe and its parameters. The scale set by this ruler is the sound horizon at the time of recombination. This is an epoch where neutral atoms formed and the universe became transparent to photons. The CMB provides a snapshot of this epoch, which occure when the universe was ~400,000 years old. The sound horizon at this time determines the location of the first acoustic peak in the CMB power spectrum, and also sets a scale associated with baryon acoustic oscillations. We will discuss both of these effects in detail later on.

Galaxy Counts:

Finally, we can also find H_0 and q_0 by counting the number of galaxies as a function of redshift. The number of galaxies at a slice between r and $r+dr$ ("r" being the comoving coordinate of that slice) is given by:

$$dN_g = n_g \, dV = n_g \, a^3_{(t)} \, r_1^2 \, dr_1 \, d\Omega$$



We assume flat geometry again, which is valid in the local universe

Assuming a constant number for galaxies that is subject to expansion only, we have:

$n_g a^3_{(t)} = n_0 a^3_{(t_0)}$ n_0: number dencity of galaxies at the present time $(z=0)$

Therefore:

$$dN_g = n_0 a^3_{(t_0)} r_1^2 dr_1 d\Omega$$

By using the relation:

$$r_1 = H_0^{-1} a^{-1}_{(t_0)} \left[z - \frac{1}{2} (1+q_0) z^2 \right]$$

We can express $r_1^2 dr_1$ in terms of $z^2 dz$:

$$d_r = H_0^{-1} a^{-1}_{(t_0)} \left[1 - (1+q_0) z \right] dz$$

The final results is:

$$\frac{dN_g}{dz\, d\Omega} = \frac{z^2 n_0}{H_0^3} \left[1 - 2 (1+q_0) z \right]$$

From this expression, we can find H_0 and q_0 after measuring N_g as a function of redshift z.